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Multipliers for the Generalized Riemann Integral*

JEFF W. MORTENSEN

Department of Mathematics, University of Nevada, Reno, Nevada 89557

AND

WASHEK F. PFEFFER

Department of Mathematics, University of California, Davis, California 95616

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The Stokes theorem for noncontinuously differentiable forms has been established by means of a coordinate free Riemann-type integral, which integrates the divergence of any differentiable vector field over bounded sets of finite perimeter. We show that the pointwise products of integrable and Lipschitz functions are integrable, and interpret the integrable functions as distributions.

For an integer $m \geq 1$, the m-fold Cartesian product of the set $\mathbf{R}$ of all real numbers is denoted by $\mathbf{R}^m$. The metric in $\mathbf{R}^m$ is induced by the norm $|x| = \max\{|\xi_1|, \ldots, |\xi_m|\}$. Thus the set $U(x, r) = \{y \in \mathbf{R}^m : |x - y| < r\}$ is an open cube of diameter $2r$. If $E \subset \mathbf{R}^m$, then $d(E)$, cl E, int E, and ∂E denote, respectively, the diameter, closure, interior, and boundary of E. The k-dimensional (outer) Hausdorff measure $\mathcal{H}^k$ in $\mathbf{R}^m$ is defined so that it agrees with the k-dimensional Lebesgue measure in $\mathbf{R}^k \subset \mathbf{R}^m$ for $k = 1, \ldots, m$; as usual, $\mathcal{H}^0$ is the counting measure in $\mathbf{R}^m$. A set $E \subset \mathbf{R}^m$ with $\mathcal{H}^k(E) = 0$ is called *$\mathcal{H}^k$-negligible*.

Let $E \subset \mathbf{R}^m$. We say that $x \in \mathbf{R}^m$ is, respectively, a *density* or *dispersion* point of E whenever

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$$\liminf_{r\to 0+} \frac{\mathcal{H}^m(E \cap U(x, r))}{\mathcal{H}^m(U(x, r))} = 1 \qquad \text{or} \qquad \limsup_{r\to 0+} \frac{\mathcal{H}^m(E \cap U(x, r))}{\mathcal{H}^m(U(x, r))} = 0.$$

The set of all density points of E is called the *essential interior* of E, denoted by int^*E, and the set of all nondispersion points of E is called the *essential closure* of E, denoted by cl^*E. The *essential boundary* of E is the set $\partial^*E = \mathrm{cl}^*E - \mathrm{int}^*E$. Clearly $\mathrm{int}\, E \subset \mathrm{int}^*E \subset \mathrm{cl}^*E \subset \mathrm{cl}\, E$, and so $\partial^*E \subset \partial E$. By [S, Chap. IV, Theorem 4.2], the sets int^*E, cl^*E, and ∂^*E are $\mathcal{H}^m$-measurable. If $E = \mathrm{cl}^*E$, we say that E is *essentially closed.*

A bounded set $A \subset \mathbf{R}^m$ is called a BV set (BV for *bounded variation*) whenever the number $\|A\| = \mathcal{H}(\partial^*A)$, called the *perimeter* of A, is finite. By [Fe, Section 2.10.6 and Theorem 4.5.11], the family BV_m of all BV sets coincides with the collection of all bounded measurable subsets of $\mathbf{R}^m$ whose De Giorgi perimeters are finite (cf. [MM, Section 2.1.2]). Since

$$\max\{\|A \cup B\|, \|A \cap B\|, \|A - B\|\} \le \|A\| + \|B\|$$

for all $A, B \in \mathrm{BV}_m$, the family BV_m is closed with respect to finite unions and intersections, and set differences. The *regularity* of a BV set A is the number

$$r(A) = \begin{cases} \dfrac{\mathcal{H}^m(A)}{d(A)\|A\|} & \text{if } d(A)\|A\| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $E \subset \mathbf{R}^m$. A nonnegative function δ defined on cl^*E is called a *gage* in E whenever the set $\{x \in \mathrm{cl}^*E : \delta(x) = 0\}$ has σ-finite measure $\mathcal{H}^{m-1}$. A *partition* in E is a collection $P = \{(A_1, x_1), \ldots, (A_p, x_p)\}$ where $A_1, \ldots, A_p$ are disjoint BV subsets of E and $x_i \in \mathrm{cl}^*A_i$ for $i = 1, \ldots, p$. Given $\varepsilon > 0$ and a gage δ in E, we say that P is *ε-regular* or *δ-fine* if $r(A_i) > \varepsilon$ for $i = 1, \ldots, p$ or $d(A_i) < \delta(x_i)$ for $i = 1, \ldots, p$, respectively.

All the functions we consider are real-valued. The algebraic operations, order, and convergence among functions on the same set are defined pointwise. If f is a function on a set A and $B \subset A$, we denote by $f \restriction B$ the restriction of f to B; when no confusion can arise, we write f instead of $f \restriction B$. A function F defined on a family $\mathcal{C} \subset \mathrm{BV}_m$ is called

(1) *additive* if $F(\cup\mathcal{D}) = \sum_{D\in\mathcal{D}} F(D)$ for each finite disjoint collection $\mathcal{D} \subset \mathcal{C}$ whose union belongs to $\mathcal{C}$.

(2) *continuous* is given $\varepsilon > 0$, there is an $\eta > 0$ such that $|F(C)| < \varepsilon$ for each $C \in \mathcal{C}$ with $\|C\| < 1/\varepsilon$ and $\mathcal{H}^m(C) < \eta$.

According to [P2, Proposition 7.7] and [P3, Theorem 3.3], the following definition of integrability is equivalent to those introduced in [P2, Definition 5.1] and [P3, Definition 3.1].

DEFINITION. Let $A \in \mathrm{BV}_m$ and let f be a function defined on cl^*A. We say that f is *integrable* in A if there is an additive continuous function F defined on the family of all BV subsets of A and having the following property: given $\varepsilon > 0$, we can find a gage δ in A so that

$$\sum_{i=1}^{p} |f(x_i)\mathcal{H}^m(A_i) - F(A_i)| < \varepsilon$$

for each δ-fine ε-regular partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A.

It follows from [P2, Corollaries 5.5 and 5.9] that the function F of the above definition, called the *indefinite integral* of f in A, is determined uniquely by the restriction $f \restriction (\mathrm{cl}^*A - N)$ where N is any $\mathcal{H}^m$-negligible set: in particular, F is determined uniquely by $f \restriction A$ (see [S, Chap. IV, Theorem 6.1]). For each BV set $B \subset A$, we let $\int_B f\,d\mathcal{H}^m = F(B)$ and call this number the *integral* of f over B. This notation leads to no confusion, as it is clear that $\int_B f\,d\mathcal{H}^m = \int_B (f \restriction B)\,d\mathcal{H}^m$. The family of all integrable functions in A is denoted by $\mathcal{I}(A)$.

A *multiplier* for a family $\mathcal{E}$ of functions on a set E is a function g on E such that $fg \in \mathcal{E}$ for every $f \in \mathcal{E}$. We show that all Lipschitz functions on a set $A \in \mathrm{BV}_m$ are multipliers for the family $\mathcal{I}(A)$. If $m = 1$, this follows from the integration by parts theorem established in [BGP, Corollary 4.3]; however, a new technique is required if $m > 1$. Before presenting the formal proof, we describe its main idea.

If $C \subset \mathbf{R}^{m+1}$ and $t \in \mathbf{R}$, let $C^t = \{x \in \mathbf{R}^m : (x, t) \in C\}$. Suppose that a Lipschitz function g on a set $A \in \mathrm{BV}_m$ maps A into the unit interval $I = [0, 1]$, and observe that for each BV set $B \subset A$, the subgraph $\Sigma_B = \{(x, t) \in B \times I : t \leq g(x)\}$ of g belongs to BV_{m+1}. Choose an $f \in \mathcal{I}(A)$, and let $(f \otimes 1)(x, t) = f(x)$ for every $(x, t) \in A \times I$. Assuming that $f \otimes 1$ belongs to $\mathcal{I}(A \times I)$ and applying formally Fubini's theorem, we obtain

$$\begin{aligned}\int_B fg\,d\mathcal{H}^m &= \int_B \left[f(x)\int_0^{g(x)} d\mathcal{H}^1(t)\right] d\mathcal{H}^m(x) = \int_{\Sigma_B} f \otimes 1\,d\mathcal{H}^{m+1} \\ &= \int_I \left[\int_{(\Sigma_B)^t} f(x)\;\; d\mathcal{H}^m(x)\right] d\mathcal{H}^1(t).\end{aligned}$$

Although the Fubini theorem does not hold for the integral under consideration (see [P1, Example 5.7]), we show that the function

$$B \mapsto \int_I \left[\int_{(\Sigma_B)^t} f(x)\,d\mathcal{H}^m(x)\right] d\mathcal{H}^1(t)$$

is still the indefinite integral of fg.

LEMMA 1. *If F is an additive continuous function defined on the family of all BV subsets of a set $A \in \mathrm{BV}_m$, then*

$$\lim_{\|B\| \to +\infty} \frac{F(B)}{\|B\|} = 0 \qquad \text{and} \qquad \sup_{\|B\| < c} |F(B)| < +\infty$$

for each $c > 0$.

Proof. Let $\mathbf{0} = (0, \ldots, 0)$ be the origin of $\mathbf{R}^m$, and find an $r > 0$ with $A \subset U(\mathbf{0}, r)$. If $a = 2^m r^{m-1}$, then the $\mathcal{H}^{m-1}$ measure of each face of $U(\mathbf{0}, r)$ equals $a/2$. Given $c \geq 1$, choose a positive $\varepsilon < 1/[c(1 + 2a)]$ and find an $\eta > 0$ so that $|F(B)| < \varepsilon$ for each BV set $B \subset A$ with $\|B\| < 1/\varepsilon$ and $\mathcal{H}^m(B) < \eta$. Select an integer $p > (2r)^m/\eta$ and, for $i = 1, \ldots, p$, let

$$A_i = \left[-r + (i - 1)\frac{2r}{p}, -r + i\frac{2r}{p}\right) \times [-r, r]^{m-1}.$$

If C is a BV subset of A and $\|C\| < c$, then

$$\mathcal{H}^m(C \cap A_i) \leq \mathcal{H}^m(A_i) < \eta \qquad \text{and} \qquad \|C \cap A_i\| \leq \|C\| < c \leq \frac{1}{\varepsilon}$$

for $i = 1, \ldots, p$. Thus $|F(C)| \leq \sum_{i=1}^{p} |F(C \cap A_i)| < p\varepsilon$, which proves the second claim.

To prove the first claim, choose a BV set $C \subset A$ so that $\|C\| > \max\{p, 1/\varepsilon\}$, and for each $t \in (-r, r)$, let

$$C_-(t) = C \cap ([-r, t) \times [-r, r]^{m-1})$$

and

$$C_+(t) = C \cap ([t, r] \times [-r, r]^{m-1}).$$

Then C is the disjoint union of BV sets $C_\pm(t)$, and it is easy to see that

$$\|C_-(t)\| + \|C_+(t)\| \leq \|C\| + a.$$

Observe that $t \mapsto \|C_-(t)\|$ is an increasing function on $(-r, r)$, which raises from 0 to $\|C\|$. Since

$$\lim_{t \to \tau+} \|C_-(t)\| - \lim_{t \to \tau-} \|C_-(t)\| \leq a$$

for each $\tau \in (-r, r)$, there is a $\theta \in (-r, r)$ such that

$$\frac{\|C\|}{2} < \|C_-(\theta)\| \leq \frac{\|C\|}{2} + a.$$

We conclude that $\|C_{\pm}(\theta)\| \leq \|C\|/2 + a$. Next find an integer $n \geq 1$ with

$$\frac{\|C\|}{2^n} < \frac{1}{\varepsilon} - 2a \leq \frac{\|C\|}{2^{n-1}},$$

and proceeding inductively, construct disjoint BV sets $C_1, \ldots, C_{2^n}$ whose union is C and such that

$$\|C_k\| \leq \frac{\|C\|}{2^n} + \sum_{j=0}^{n-1} \frac{a}{2^j} < \frac{\|C\|}{2^n} + 2a < \frac{1}{\varepsilon}.$$

Note that the inequality $1 < 1/\varepsilon - 2a$ yields $2^{n-1} < \|C\|$. For $i = 1, \ldots, p$ and $k = 1, \ldots, n$, we have $|A_i \cap C_k| \leq |A_i| < \eta$ and $\|A_i \cap C_k\| \leq \|C_k\| < 1/\varepsilon$. By construction, the collection

$$\{A_i \cap C_k : i = 1, \ldots, p; k = 1, \ldots, 2^n\}$$

contains at most $2^n + p - 1$ nonempty sets whose union is C. Therefore

$$|F(C)| \leq \sum_{i=1}^{p} \sum_{k=1}^{2^n} |F(A_i \cap C_k)| < \varepsilon(2^n + p - 1) < \varepsilon(2\|C\| + p),$$

and hence

$$\frac{|F(C)|}{\|C\|} < \varepsilon\left(2 + \frac{p}{\|C\|}\right) < 3\varepsilon.$$

PROPOSITION 1. *An additive function F defined on the family of all BV subsets of a set $A \in \mathrm{BV}_m$ is continuous if and only if the following condition is satisfied: given $\varepsilon > 0$, there is a $\theta > 0$ such that*

$$|F(B)| < \theta \mathcal{H}^m(B) + \varepsilon(\|B\| + 1)$$

for each BV set $B \subset A$.

Proof. As the converse is obvious, assume that F is continuous and choose an $\varepsilon > 0$. According to Lemma 1, there are positive numbers b and c such that $|F(B)| < \varepsilon\|B\|$ and $|F(C)| < b$ whenever $B, C \in \mathrm{BV}_m$ are such that $\|B\| \geq c$ and $\|C\| < c$. We can find an $\eta > 0$ so that $|F(C)| < \varepsilon$ for each $C \in \mathrm{BV}_m$ for which $\|C\| < c$ and $\mathcal{H}^m(C) < \eta$. Now if $\|C\| < c$ and $\mathcal{H}^m(C) \geq \eta$, then $|F(C)| < b \leq (b/\eta)\mathcal{H}^m(C)$. Letting $\theta = b/\eta$, the previous alternatives yield the desired inequality.

By [W, Corollary 10.9], there exists a finitely additive function λ defined on the family of all subsets of $\mathbf{R}$ such that $\lambda(A) = \mathcal{H}^1(A)$ for each $\mathcal{H}^1$-

measurable set $A \subset \mathbf{R}$; in fact, λ can be assumed translation invariant, but we shall not need this. Given an additive function F on the family of all BV subsets of a set $A \in \mathrm{BV}_m$, we let

$$\hat{F}(C) = \int_{\mathbf{R}} F(C^t)\, d\lambda(t)$$

for each BV set $C \subset A \times \mathbf{R}$. Note that $\hat{F}(C)$ is a well-defined real number. Indeed, it follows from [M, Theorems 20 and 33] that $C^t \in \mathrm{BV}_m$ for $\mathcal{H}^1$-almost all $t \in \mathbf{R}$, and $C^t = \varnothing$ for all sufficiently large $|t|$. Since λ is only finitely additive, the integral used in the definition of $\hat{F}(C)$ is just a nonnegative linear functional, which does not have the usual continuity properties corresponding to the monotone and dominated convergence theorems.

LEMMA 2. *If F is an additive continuous function on the family of all BV subsets of a set $A \in \mathrm{BV}_m$, then $\hat{F}$ is an additive continuous function on the family of all BV subsets of $A \times [0, 1]$.*

Proof. As the additivity of $\hat{F}$ is clear, choose an $\varepsilon > 0$ and find a $\theta > 0$ associated with F and ε according to Proposition 1. If $C \subset A \times [0, 1]$ is a BV set, [MM, Section 2.2.1, inequality (19), p. 85] yields

$$\begin{aligned}|\hat{F}(C)| &\le \int_0^1 |F(C^t)|\, d\lambda < \theta \int_0^1 \mathcal{H}^m(C^t)\, d\lambda + \varepsilon \int_0^1 (1 + \|C^t\|)\, d\lambda \\ &\le \theta \mathcal{H}^{m+1}(C) + \varepsilon(1 + \|C\|),\end{aligned}$$

which implies the continuity of $\hat{F}$.

Remark 1. The function λ has been employed merely for convenience. If C is a *figure* (i.e., a finite union of intervals), then $\hat{F}(C) = \int_{\mathbf{R}} F(C^t)\, d\mathcal{H}^1$ since $t \mapsto F(C^t)$ is a simple $\mathcal{H}^1$-measurable function. As the general BV sets can be approximated by figures (see [G, Theorem 1.24] and [NP, Lemma 3.1]), the additivity and continuity of F can be used to define $\hat{F}$ (cf. [NP, Proposition 3.2]). In particular, $\hat{F}$ depends only on F, and not on the choice of λ. To prove this directly by showing that the function $t \mapsto F(C^t)$ is $\mathcal{H}^1$-measurable for each BV set C is possible but long.

THEOREM. *Let g be a Lipschitz function on a set $A \in \mathrm{BV}_m$. If f belongs to $\mathcal{I}(A)$, then so does fg.*

Proof. Avoiding a triviality, suppose that $\mathcal{H}^m(A) > 0$. The function g determines uniquely a Lipschitz function on cl^*A, still denoted by g. Since g is bounded and $\mathcal{I}(A)$ is a linear space containing the constant functions, we may assume that g maps cl^*A into the unit interval $I = [0, 1]$. Choose a $\gamma \in \mathbf{R}$ larger than the Lipschitz constant of g, and let $c = 1 +$

$\sqrt{1+\gamma^2}$. For each BV set $B \subset A$, the set

$$\Sigma_B = \{(x, t) \in B \times I : t \leq g(x)\}$$

belongs to BV_{m+1}; for $\|\Sigma_B\| \leq c\mathcal{H}^m(B) + \|B\| < +\infty$ by [G, Theorem 14.6]. Select an $f \in \mathcal{J}(A)$ and denote by F the indefinite integral of f in A. By Lemma 2, the map $G : B \mapsto \hat{F}(\Sigma_B)$ is an additive continuous function on the family of all BV subsets of A. Using a technique similar to that of [J, Theorem 6.8], we show that G is an indefinite integral of fg.

To this end, choose an $\varepsilon > 0$ and find an $\eta > 0$ so that $|\hat{F}(C)| < \varepsilon$ for each BV set $C \subset A \times I$ with $\|C\| < (c + \gamma/\varepsilon)\mathcal{H}^m(A)$ and $\mathcal{H}^{m+1}(C) < \eta[\gamma\mathcal{H}^m(A)]$. There is a gage δ on A such that

$$\sum_{i=1}^{p} |f(x_i)\mathcal{H}^m(A_i) - F(A_i)| < \varepsilon$$

for each δ-fine ε-regular partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A. With no loss of generality, we may assume that $\delta \leq \eta$. Let $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ be a δ-fine ε-regular partition in A, and let $J_i = [0, g(x_i)]$ for $i = 1, \ldots, p$. We obtain

$$\sum_{i=1}^{p} |f(x_i)g(x_i)\mathcal{H}^m(A_i) - G(A_i)| \leq \sum_{i=1}^{p} g(x_i)|f(x_i)\mathcal{H}^m(A_i) - F(A_i)|$$

$$+ \sum_{i=1}^{p} |F(A_i)\lambda(J_i) - \hat{F}(\Sigma_{A_i})| < \varepsilon + \sum_{i=1}^{p} |\hat{F}(A_i \times J_i) - \hat{F}(\Sigma_{A_i})|.$$

If $S = \sum_{i=1}^{p} |\hat{F}(A_i \times J_i) - \hat{F}(\Sigma_{A_i})|$, then after a suitable reordering we find an integer k with $0 \leq k \leq p$ such that

$$\begin{aligned} S &= \left|\sum_{i=1}^{k} \left[\hat{F}(A_i \times J_i) - \hat{F}(\Sigma_{A_i})\right]\right| + \left|\sum_{i=k+1}^{p} \left[\hat{F}(A_i \times J_i) - \hat{F}(\Sigma_{A_i})\right]\right| \\ &= \left|\hat{F}\left[\bigcup_{i=1}^{k} (A_i \times J_i)\right] - \hat{F}\left[\bigcup_{i=1}^{k} \Sigma_{A_i}\right]\right| \\ &\quad + \left|\hat{F}\left[\bigcup_{i=k+1}^{p} (A_i \times J_i)\right] - \hat{F}\left[\bigcup_{i=k+1}^{p} \Sigma_{A_i}\right]\right| \\ &\leq \left|\hat{F}\left[\bigcup_{i=1}^{k} (A_i \times J_i - \Sigma_{A_i})\right]\right| + \left|\hat{F}\left[\bigcup_{i=1}^{k} (\Sigma_{A_i} - A_i \times J_i)\right]\right| \\ &\quad + \left|\hat{F}\left[\bigcup_{i=k+1}^{p} (A_i \times J_i - \Sigma_{A_i})\right]\right| + \left|\hat{F}\left[\bigcup_{i=k+1}^{p} (\Sigma_{A_i} - A_i \times J_i)\right]\right|. \end{aligned}$$

We let $C = \bigcup_{i=1}^{k} (A_i \times J_i - \Sigma_{A_i})$, and estimate $|\hat{F}(C)|$ by observing that

$$\mathcal{H}^{m+1}(A_i \times J_i - \Sigma_{A_i}) \leq \gamma d(A_i)\mathcal{H}^m(A_i) < \eta[\gamma\mathcal{H}^m(A_i)],$$
$$\|A_i \times J_i - \Sigma_{A_i}\| \leq c\mathcal{H}^m(A_i) + \gamma d(A_i)\|A_i\| < \left(c + \frac{\gamma}{\varepsilon}\right)\mathcal{H}^m(A_i)$$

for $i = 1, \ldots, k$ (see [G, Theorem 14.6]). Indeed, these estimates imply that

$$\mathcal{H}^{m+1}(C) < \eta[\gamma\mathcal{H}^m(A)] \qquad \text{and} \qquad \|C\| < \left(c + \frac{\gamma}{\varepsilon}\right)\mathcal{H}^m(A),$$

and consequently $|\hat{F}(C)| < \varepsilon$. Completely analogous verifications show that $S < 4\varepsilon$, and the theorem is proved.

Remark 2. If g is a BV function defined in an open set containing cl^*A (see [G, Definition 1.3]), then [G, Theorem 14.6] still implies that Σ_B belongs to BV_{m+1}. Thus the continuous additive function G of the previous proof is well-defined, and it is conceivable that by using more refined estimates one may still show that G is the indefinite integral of fg.

A sequence $\{g_n\}$ of Lipschitz functions defined on a set $E \subset \mathbf{R}^m$ is called *equilipschitz* whenever the Lipschitz constant of each g_n is smaller than a fixed $c \in \mathbf{R}$, which does not depend on n.

LEMMA 3. *Let $\{g_n\}$ be an equilipschitz sequence of functions defined on a bounded essentially closed set $E \subset \mathbf{R}^m$. If E is nonempty, then the following conditions are equivalent*:

(i) $\lim g_n = 0$ *uniformly in* E;

(ii) $\lim g_n = 0$ $\mathcal{H}^m$*-almost everywhere in* E;

(iii) $\lim \int_E |g_n| \, d\mathcal{H}^m = 0$.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious, and since E is bounded, (ii) $\Rightarrow$ (iii) follows from the dominated convergence theorem. For completeness, we incude the proof of (iii) $\Rightarrow$ (i), which can be found in [KMP, Lemma 3.10]. Every g_n has a unique extension to the compact set $\mathrm{cl}\, E$, still denoted by g_n. Choose a $\gamma \in \mathbf{R}$ larger than the Lipschitz constant of each g_n. Proceeding towards a contradiction, suppose that there is an $\varepsilon > 0$ such that, for $n = 1, 2, \ldots$, we can find a $z_n \in \mathrm{cl}\, E$ with $|g_n(z_n)| \geq 3\varepsilon$. Now for a $z \in \mathrm{cl}\, E$, infinitely many points z_n lie in $U = U(z, \varepsilon/\gamma)$. It follows that

$$|g_n(x)| \geq |g_n(z_n)| - \gamma|z_n - x| > 3\varepsilon - \gamma(|z_n - z| + |z - x|) > \varepsilon$$

for infinitely many n and all $x \in E \cap U$. Since E is essentially closed,

$$\lim \int_E |g_n|\, d\mathcal{H}^m \geq \limsup \int_{E \cap U} |g_n|\, d\mathcal{H}^m \geq \varepsilon \mathcal{H}^m(E \cap U) > 0,$$

a contradiction.

PROPOSITION 2. *Let $\{g_n\}$ be an equilipschitz sequence of functions defined on a set $A \in \mathrm{BV}_m$, and let f belong to $\mathcal{I}(A)$. If* $\lim g_n = 0$ *almost everywhere in A, then* $\lim \int_A fg_n \, d\mathcal{H}^m = 0$.

Proof. Each g_n determines uniquely a Lipschitz function on cl^*A, still denoted by g_n. By Lemma 3, there is a sequence $\{\varepsilon_n\}$ of positive constants such that $\lim \varepsilon_n = 0$ and $|g_n(x)| \leq \varepsilon_n$ for $n = 1, 2, \ldots$ and each $x \in \mathrm{cl}^*A$. The sequence $\{g_n + \varepsilon_n\}$ is equilipschitz, and $0 \leq g_n + \varepsilon_n \leq 2\varepsilon_n \leq 1$ for all sufficiently large n. Since

$$\lim \int_A fg_n \, d\mathcal{H}^m = \lim \int_A f(g_n + \varepsilon_n)\, d\mathcal{H}^m$$

whenever either limit exists, we may assume that each g_n maps cl^*A into the unit interval $I = [0, 1]$. Choose a $\gamma \in \mathbf{R}$ larger than the Lipschitz constant of each g_n, and let $c = 1 + \sqrt{1 + \gamma^2}$. Following the proof of the theorem above, it is easy to see that the sets $\Sigma_n = \{(x, t) \in A \times I : t \leq g_n(x)\}$ belong to BV_{m+1}, and that $\mathcal{H}^{m+1}(\Sigma_n) \leq \varepsilon_n \mathcal{H}^m(A)$ and $\|\Sigma_n\| \leq c\mathcal{H}^m(A) + \|A\|$ for $n = 1, 2, \ldots$ Moreover, if F is the indefinite integral of f in A, then $\int_A fg_n \, d\mathcal{H}^m = \hat{F}(\Sigma_n)$. In view of the previous estimates, the proposition follows from Lemma 2.

COROLLARY. *Let $A \in \mathrm{BV}_m$ and $f \in \mathcal{I}(A)$. Set $\Lambda(\varphi) = \int_A f\varphi \, d\mathcal{H}^m$ for each C^∞ function φ on $\mathbf{R}^m$ vanishing outside a compact set. Then Λ is a distribution in $\mathbf{R}^m$ of order at most one whose support is contained in* $\mathrm{cl}\, A$.

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